A Parallel Iterative System Solver

Nikolaos M. Missirlis
Department of Applied Mathematics
Athens University, Panepistemiopolis
Athens 157 00, Greece

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ABSTRACT

We study the adoption of iterative methods for numerically solving linear systems of the form $Au = b$ on parallel machines. A new class of first order iterative schemes possessing a high level of parallelism is originated by the approximation of the Neumann series to A^{-1}. A preliminary study of the case where the sequence of parameters involved are constant and equal to unity reveals that the series is best approximated by its first two terms. This results in the derivation of a new iterative method which under certain conditions possesses an exceptionally high rate of convergence.

1. INTRODUCTION

Although there exist various parallel algorithms which solve linear systems using direct methods (see e.g. [13], [9], [4]), the application of iterative schemes has not aroused similar enthusiasm. In an attempt to stimulate further research [2, 1] in this area, a new class of first order iterative methods, suitable for parallel implementation on SIMD and MIMD machines [6], is derived in the present paper (see also [3], [8], [11]). Let us commence our study by considering the numerical solution of linear systems of the form

$$Au = b, \tag{1.1}$$

where it is assumed throughout that A is a given real, nonsingular matrix of order N, with nonvanishing diagonal elements, and b, u are two N-dimensional vectors with b given and u to be determined. By expressing the coefficient matrix A in the form

$$A = D - C_L - C_U, \tag{1.2}$$

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where $D = \mathrm{diag}(A)$, and $-C_L$, $-C_U$ are the strictly lower and upper triangular parts of A, respectively, then (1.2) can be written alternatively as

$$A = D(I - B), \tag{1.3}$$

where

$$B = L + U, \qquad L = D^{-1}C_L, \quad \text{and} \quad U = D^{-1}C_U. \tag{1.4}$$

Next, we determine the inverse of A using (1.3). Thus, by assuming that the spectral radius $S(B)$ of B is less than unity, we have

$$A^{-1} = \left(\sum_{k=0}^{\infty} B^k \right) D^{-1}. \tag{1.5}$$

An approximation of the infinite series (1.5) by keeping the first l terms yields

$$A_l^{-1} = \left(\sum_{k=0}^{l-1} \omega_k B^k \right) D^{-1}, \tag{1.6}$$

where ω_k, $k = 0(1)l - 1$, is a sequence of real parameters. The determination of the sequence ω_k, $k = 0(1)l - 1$, will be such that (1.6) is the best possible approximation of (1.5). The expression for A_l^{-1} offers an attractive choice for the inverse of the conditioning matrix R in

$$u^{(n+1)} = u^{(u)} + \tau R^{-1}(b - Au^{(n)}), \tag{1.7}$$

where τ ($\neq 0$) is a real parameter. As is usual in our work [5, 10], (1.7) is used for the construction of a first order iterative method for solving (1.1) numerically. Evidently, by letting $R^{-1} = A_l^{-1}$, Equation (1.7), because of (1.6), yields the following class of first order iterative methods:

$$u^{(n+1)} = u^{(n)} + \tau \left(\sum_{k=0}^{l-1} \omega_k B^k \right) D^{-1}(b - Au^{(n)}), \qquad l = 1,2,3,\ldots, \tag{1.8}$$

which alternatively can be written as

$$u^{(n+1)} = \Gamma_{\tau,\omega_k} u^{(n)} + \gamma_{\tau,\omega_k}, \qquad l = 1,2,3,\ldots, \tag{1.9}$$

where

$$\Gamma_{\tau,\omega_k} = I - \tau\mathscr{B}_{\omega_k}, \qquad \gamma_{\tau,\omega_k} = \tau\left(\sum_{k=0}^{l-1} \omega_k B^k\right) D^{-1}b,$$
$$\mathscr{B}_{\omega_k} = \left(\sum_{k=0}^{l-1} \omega_k B^k\right) D^{-1}A = \sum_{k=0}^{l} (\omega_k - \omega_{k-1}) B^k \tag{1.10a}$$

with

$$\omega_0 = 1 \quad \text{and} \quad \omega_{-1} = \omega_l = 0. \tag{1.10b}$$

A simple examination of (1.8) reveals that the defined iterative procedures possess a high level of inherent parallelism and therefore are suitable for parallel implementation on SIMD and MIMD machines. However, we note that as l increases, the amount of work involved in each iteration is prohibitively large. The first question which therefore arises is: What is the optimum value of l in the sense of saving superfluous computational effort? An answer to this question is suggested in Section 2 in the case where $\omega_k = 1$, $k = 0(1)l-1$. In Section 3 we develop the convergence analysis of the iterative scheme which is obtained from (1.9) when $\omega_k = \omega$, $k = 0(1)l-1$, and $l = 2$. Finally, our remarks and conclusions are summarized in Section 4.

2. THE PARALLEL l-FOLD JACOBI METHOD

As was mentioned in the introduction, the choice of the number of terms in (1.6) for a good approximation of A^{-1} has a serious influence on the efficiency of (1.9). In the present section we will attempt to find the value of l which reduces the spectral radius of $\Gamma_{\tau,1}$ [i.e. Γ_{τ,ω_k}, when $\omega_k = 1$, $k = 1(1)l-1$] with the lowest computational cost. For the choice $\omega_k = 1$, $k = 1(1)l-1$, (1.9) and (1.10) yield successively

$$u^{(n+1)} = \Gamma_{\tau,1}u^{(n)} + \gamma_{\tau,1}, \qquad l = 1,2,\ldots, \tag{2.1}$$

where

$$\Gamma_{\tau,1} = I - \tau\mathscr{B}_1, \qquad \gamma_{\tau,1} = \tau\left(\sum_{k=0}^{l-1} B^k\right) D^{-1}b, \tag{2.2}$$

and

$$\mathscr{B}_1 = \left(\sum_{k=0}^{l-1} B^k \right) D^{-1}A = (I - B^l). \tag{2.3}$$

We note that for $\tau = 1$, (2.1) is equivalent to l applications of the Jacobi method; it will be referred to as the *extrapolated l-fold Jacobi method*. In order to examine the role of l we have to determine the spectral radius $S(\Gamma_{\tau,1})$ of $\Gamma_{\tau,1}$. If λ, μ denote the eigenvalues of $\mathscr{B}_1$ and B, respectively, then by (2.3), we have the eigenvalue relationship

$$\lambda = 1 - \mu^l, \qquad l = 1,2,3,\cdots. \tag{2.4}$$

We further distinguish two cases according to whether l possesses even or odd values.

Case I: l Is Even

In this case we prove the following theorem.

THEOREM 2.1. *Let A be a matrix with nonvanishing diagonal elements such that the matrix B has real eigenvalues μ with $\underline{\mu} = \min|\mu| \neq 0$, $\bar{\mu} = \max|\mu|$, and l even. Then* (2.1) *converges iff*

$$\bar{\mu} < 1 \quad \textit{and} \quad 0 < \tau < \frac{2}{1 - \underline{\mu}^l} \tag{2.5}$$

or

$$\underline{\mu} > 1 \quad \textit{and} \quad \frac{2}{1 - \bar{\mu}^l} < \tau < 0. \tag{2.6}$$

Proof. Necessary and sufficient conditions for (2.1) to converge are [7]

$$\lambda > 0 \quad \text{and} \quad 0 < \tau < 2/\bar{\lambda}, \tag{2.7}$$

where

$$\bar{\lambda} = \max_{\underline{\mu} \leq |\mu| \leq \bar{\mu}} \lambda, \tag{2.8}$$

or

$$\lambda < 0 \quad \text{and} \quad 2/\bar{\lambda} < \tau < 0, \tag{2.9}$$

where

$$\bar{\lambda} = \min_{\underline{\mu} \leq |\mu| \leq \bar{\mu}} \lambda. \tag{2.10}$$

From (2.4), when l is even, $\lambda = 1 - \mu^l > 0$ iff $\mu^l < 1$, or equivalently

$$\bar{\mu} < 1. \tag{2.11}$$

Moreover, because of (2.8),

$$\bar{\lambda} = 1 - \underline{\mu}^l. \tag{2.12}$$

Hence (2.5) is proved. Similarly, $\lambda = 1 - \mu^l < 0$ iff $\mu^l > 1$, or equivalently,

$$\underline{\mu} > 1. \tag{2.13}$$

Since, in this case [see (2.10)]

$$\bar{\lambda} = 1 - \bar{\mu}^l, \tag{2.14}$$

then because of (2.9), (2.6) follows. ■

COROLLARY 2.2. *Under the hypothesis of Theorem* 2.1, *if* $\underline{\mu} = 0$, *then* (2.1) *converges iff*

$$\bar{\mu} < 1 \quad \textit{and} \quad 0 < \tau < 2. \tag{2.15}$$

Proof. It follows from Theorem 2.1. ■

For the optimum value of the parameter τ we have:

THEOREM 2.3. *Let A be a matrix with nonvanishing diagonal elements such that the matrix B has real eigenvalues* μ *with* $\underline{\mu} = \min |\mu| \neq 0$, $\bar{\mu} =$

$\max|\mu|$, *and* l *is even. Then* $S(\Gamma_{\tau,1})$ *is minimized for*

$$\tau_0 = \frac{2}{2-\left(\bar{\mu}^l + \underline{\mu}^l\right)}, \tag{2.16}$$

and its corresponding value is given by the expression

$$S\left(\Gamma_{\tau_0,1}\right) = \begin{cases} \tau_0\left(\bar{\mu}^l - \underline{\mu}^l\right)/2 & \text{if} \quad \bar{\mu} < 1, \\ \tau_0\left(\underline{\mu}^l - \bar{\mu}^l\right)/2 & \text{if} \quad \underline{\mu} > 1. \end{cases} \tag{2.17}$$

Proof. It is known that $S(\Gamma_{\tau,1})$ attains its minimum value, which is given by [7, 10]

$$S\left(\Gamma_{\tau_0,1}\right) = \frac{|k_l(\mathscr{B}_1) - 1|}{k_l(\mathscr{B}_1) + 1}, \tag{2.18}$$

where

$$k_l(\mathscr{B}_1) = \bar{\lambda}/\underline{\lambda}, \tag{2.19}$$

for

$$\tau_0 = \frac{2}{\bar{\lambda} + \underline{\lambda}}. \tag{2.20}$$

First we assume that $\lambda > 0$, implying $\bar{\mu} < 1$. Then by (2.4) it follows that

$$\bar{\lambda} = 1 - \underline{\mu}^l \quad \text{and} \quad \underline{\lambda} = 1 - \bar{\mu}^l. \tag{2.21}$$

Evidently, (2.20) and (2.21) yield (2.16), whereas (2.18), (2.19), and (2.21) produce the upper branch of (2.17). Next, if $\lambda < 0$, the roles of $\bar{\lambda}, \underline{\lambda}$ are interchanged and the proof of (2.17) is easily completed. ■

COROLLARY 2.4. *Under the hypothesis of Theorem* 2.3, *if* $\underline{\mu} = 0$, *then* $S(\Gamma_{\tau,1})$ *is minimized for*

$$\tau_0 = \frac{2}{2 - \bar{\mu}^l}, \tag{2.22}$$

and its corresponding value is given by

$$S(\Gamma_{\tau_0,1}) = \frac{\tau_0 \bar{\mu}^l}{2} = \frac{\bar{\mu}^l}{2 - \bar{\mu}^l}. \tag{2.23}$$

Proof. It follows from Theorem 2.3. ■

COROLLARY 2.5. *Under the hypothesis of Theorem* 2.3, *if* $\bar{\mu} = \underline{\mu} = \mu$, *then* (2.1) *converges iff either* $\mu < 1$ *and* $0 < \tau < 2/(1 - \mu^l)$ *or* $\mu > 1$ *and* $2/(1 - \mu^l) < \tau < 0$. *Moreover, we have that if*

$$\tau_0 = \frac{1}{1 - \mu^l}, \tag{2.24}$$

then

$$S(\Gamma_{\tau_0,1}) = 0. \tag{2.25}$$

Proof. It follows from Theorems 2.1 and 2.3. ■

Case II: l Is Odd

THEOREM 2.6. *Let A be a matrix with nonvanishing diagonal elements such that the matrix B has real eigenvalues* μ *with* $m \leqslant \mu \leqslant M$, *where* m, M *are the smallest and largest eigenvalues of B, respectively. If l is odd, then* (2.1) *converges iff*

$$M < 1 \quad \text{and} \quad 0 < \tau < \frac{2}{1 - m^l}. \tag{2.26}$$

Proof. Since the diagonal elements and hence the trace of B vanish [see (1.4)], it follows that the sum of the eigenvalues of B vanishes also; hence,

$$m \leqslant 0 \leqslant M. \tag{2.27}$$

On the other hand, (2.4) yields

$$\bar{\lambda} = 1 - m^l > 0 \quad \text{and} \quad \underline{\lambda} = 1 - M^l; \tag{2.28}$$

thus for convergence we must also have $\underline{\lambda} = 1 - M^l > 0$ [7] or equivalently $M < 1$. Finally, because of (2.28), (2.7) yields (2.26). ∎

COROLLARY 2.7. *Under the hypothesis of Theorem* 2.6, *if A is consistently ordered* [15], *then* (2.1) *converges iff*

$$M < 1 \quad \textit{and} \quad 0 < \tau < \frac{2}{1 + M^l}. \tag{2.29}$$

Proof. When A is consistently ordered, then $M = -m$ and (2.29) follows from (2.26). ∎

Finally, the optimum value for the parameter τ is given by the following theorem.

THEOREM 2.8. *Let A be a matrix with nonvanishing diagonal elements such that B has real eigenvalues* μ *with* $m \leqslant \mu \leqslant M < 1$, *where m, M are the smallest and largest eigenvalues of B, respectively. If l is odd, then* $S(\Gamma_{\tau,l})$ *is minimized for*

$$\tau_0 = \frac{2}{2 - M^l - m^l}, \tag{2.30}$$

and its corresponding value is given by

$$S(\Gamma_{\tau_0,l}) = \tau_0 \frac{M^l - m^l}{2} = \frac{M^l - m^l}{2 - M^l - m^l} < 1. \tag{2.31}$$

Proof. It is analogous to the proof of Theorem 2.3. ∎

For the special values of $l = 1$ and $l = 2$ the above theorems and corollaries give known results [10, 15]. However, our intention is to analyse the role of l in the spectral radius of $\Gamma_{\tau_0,l}$. As l increases the amount of computational work involved in each iteration is increased considerably (if $l = 2$ the compu-

tations are doubled, etc.). Let us therefore consider the cases where A is a consistently ordered matrix and l attains the successive values 2 and 3. Then assuming that $\underline{\mu} = 0$, we have

$$k_2(\mathscr{B}_1) = \frac{1}{1-M^2} \quad \text{and} \quad k_3(\mathscr{B}_1) = \frac{1+M^3}{1-M^3}. \tag{2.32}$$

By comparing the quantities $k_2(\mathscr{B}_1)$ and $k_3(\mathscr{B}_1)$, we find

$$\begin{aligned} \operatorname{sign}\left[k_2(\mathscr{B}_1) - k_3(\mathscr{B}_1)\right] &= \operatorname{sign}(M^3 - 2M + 1) \\ &= \operatorname{sign}\left[(M-1)(M^2+M-1)\right] \\ &= \operatorname{sign}(M^2+M-1), \end{aligned}$$

which implies that if $M \leqslant 0.6180$, then $k_3(\mathscr{B}_1) \leqslant k_2(\mathscr{B}_1)$, otherwise $k_2(\mathscr{B}_1) < k_3(\mathscr{B}_1)$. On the other hand, if we let $l = 4$, then $k_4(\mathscr{B}_1) < k_2(\mathscr{B}_1)$. Therefore, it is conjectured that it is preferable for l to attain even values, a fact which was also confirmed by numerical experiments in [3]. In fact, we let $l = 2$, since for $l = 4$ the computational cost becomes prohibitively large.

3. THE EXTRAPOLATED ω-DOUBLE JACOBI (Eω-DOJ) METHOD

In this section, we consider the iterative scheme which is derived from (1.9), (1.10), for $\omega_1 = \omega$ and $l = 2$, i.e.

$$u^{(n+1)} = \Gamma_{\tau,\omega} u^{(n)} + \gamma_{\tau,\omega}, \tag{3.1}$$

where

$$\begin{aligned} &\Gamma_{\tau,\omega} = I - \tau\mathscr{B}_\omega, \qquad \gamma_{\tau,\omega} = \tau(I+\omega B)D^{-1}b \\ &\mathscr{B}_\omega = (I+\omega B)D^{-1}A = I + (\omega-1)B - \omega B^2. \end{aligned} \tag{3.2}$$

We note that for $\omega = \tau = 1$, (3.1) degenerates into

$$u^{(n+1)} = \Gamma_{1,1} u^{(n)} + \gamma_{1,1}, \tag{3.3}$$

where

$$\Gamma_{1,1} = B^2 \quad \text{and} \quad \gamma_{1,1} = (I+B)D^{-1}b, \tag{3.4}$$

which is equivalent to two iterations of the Jacobi method. Thus (3.1) will be referred to as the *extrapolated ω-double Jacobi* (Eω-DOJ) method. In the sequel we develop the convergence analysis of the aforementioned iterative scheme under the assumption that the matrix B has two real multiple eigenvalues.

THEOREM 3.1. *Let A be a matrix with nonvanishing diagonal elements such that B has two real multiple eigenvalues m, M only, with $m < M$. Then, the Eω-DOJ method converges iff ω and τ lie in the corresponding domains of Table 1 according to the position of M and m.*

Proof. Let λ be an eigenvalue of $\mathscr{B}_\omega$. Then because of (3.2) the following eigenvalue relation holds:

$$\lambda \equiv \lambda(\mu) = (1+\omega\mu)(1-\mu). \tag{3.5}$$

A sufficient and necessary condition for the convergence of the Eω-DOJ is

$$S(\Gamma_{\tau,\omega}) = \max_{m \leqslant \mu \leqslant M} |1-\tau\lambda| < 1. \tag{3.6}$$

However, it can be readily verified that (3.6) is equivalent to either of the following set of inequalities:

Case I: $\lambda > 0$ and $0 < \tau < 2/\bar{\lambda}$, where $\bar{\lambda} = \max_{m \leqslant \mu \leqslant M} \lambda$, or
Case II: $\lambda < 0$ and $2/\bar{\lambda} < \tau < 0$, where $\bar{\lambda} = \min_{m \leqslant \mu \leqslant M} \lambda$.

TABLE 1[a]

Conditions		Case	ω-domain	τ-domain
$M<1$		I	$-1/M < \omega \leqslant \hat{\omega}$	$0 < \tau < 2/\lambda_m$
		II	$\hat{\omega} \leqslant \omega < -1/m$	$0 < \tau < 2/\lambda_M$
$M>1$	$m<1-M$	III	$-\infty < \omega < -1/M$	$0 < \tau < 2/\lambda_m$
	$m>1-M$	IV	$-\infty < \omega \leqslant \hat{\omega}$	$0 < \tau < 2/\lambda_M$
		V	$\hat{\omega} \leqslant \omega < -1/M$	$0 < \tau < 2/\lambda_m$
$M>1$	$m<1-M$	VI	$-1/m < \omega \leqslant \hat{\omega}$	$2/\lambda_M < \tau < 0$
		VII	$\hat{\omega} \leqslant \omega < +\infty$	$2/\lambda_m < \tau < 0$
	$m>1-M$	VIII	$-1/m < \omega < +\infty$	$2/\lambda_M < \tau < 0$
$M<1$		IX	$-\infty < \omega < -1/M$	$2/\lambda_m < \tau < 0$

[a] $\hat{\omega} = 1/(1-M-m)$, $\lambda_m = (1+\omega m)(1-m)$, and $\lambda_M = (1+\omega M)(1-M)$.

Case I: For this case we have that the first inequality will determine the convergence range of ω; the second, the range of τ. Because $\lambda > 0$ we must have

$$\lambda_M \equiv \lambda(M) > 0 \quad \text{and} \quad \lambda_m \equiv \lambda(m) > 0, \tag{3.7}$$

which in turn, because of (3.5) and (2.27) are equivalent to either

$$-\frac{1}{M} < \omega < -\frac{1}{m} \quad \text{and} \quad M < 1, \tag{3.8}$$

or

$$-\infty < \omega < -\frac{1}{M} \quad \text{and} \quad M > 1. \tag{3.9}$$

The relationships (3.8) or (3.9) give the convergence range of ω according to whether M is less or greater than unity. In order to explicitly determine the range for τ we have to study the behavior of λ as a function of μ for ω fixed. However, since B is assumed to possess two real eigenvalues, therefore, the possible extreme values of λ will occur at either $\mu = m$ or $\mu = M$, which implies that

$$\bar{\lambda} = \max\{\lambda_m, \lambda_M\}. \tag{3.10}$$

However,

$$\operatorname{sign}(\lambda_M - \lambda_m) = \operatorname{sign}(\omega[1 - M - m] - 1). \tag{3.11}$$

Let us first assume that $M < 1$. Then (3.11) yields

$$\operatorname{sign}(\lambda_M - \lambda_m) = \operatorname{sign}(\omega - \hat{\omega}), \tag{3.12}$$

where

$$0 < \hat{\omega} = \frac{1}{1 - M - m} < -\frac{1}{m}, \tag{3.13}$$

and because of (3.8) we have

$$\bar{\lambda} = \begin{cases} \lambda_m & \text{if} \quad -1/M < \omega \leqslant \hat{\omega}, \\ \lambda_M & \text{if} \quad \hat{\omega} \leqslant \omega < -1/m. \end{cases} \tag{3.14}$$

Alternatively, if $M > 1$, then we have to consider two further subcases according as $1 - M - m$ is positive or negative.

Subcase (i): $m \leqslant 1 - M$. In this case (3.12) holds, and since ω lies in the range given by (3.9), it follows that

$$\bar{\lambda} = \lambda_m. \tag{3.15}$$

Subcase (ii): $m > 1 - M$. In this case (3.12) yields $\operatorname{sign}(\lambda_M - \lambda_m) = \operatorname{sign}(\hat{\omega} - \omega)$, where $-\infty < \hat{\omega} < -1/M$; hence Table 2 is easily constructed.

Case II: In this case, $\lambda < 0$ is equivalent to either of the following set of inequalities:

$$-\infty < \omega < -\frac{1}{M} \quad \text{and} \quad M < 1 \tag{3.16}$$

or

$$-\frac{1}{m} < \omega < +\infty \quad \text{and} \quad M > 1. \tag{3.17}$$

By using an analogous reasoning to that in case I, we easily find that in the present case

$$\bar{\lambda} = \min\{\lambda_m, \lambda_M\}. \tag{3.18}$$

If $M < 1$, then (3.12) holds with $\hat{\omega}$ given by (3.13). Moreover, because of (3.16), $\omega < \hat{\omega}$; hence

$$\bar{\lambda} = \lambda_m. \tag{3.19}$$

Alternatively, if $M > 1$, then we consider two subcases.

TABLE 2

Condition		ω-Domain	$\bar{\lambda}$
$M > 1$	$m \leqslant 1 - M$	$-\infty < \omega < -1/M$	λ_m
	$m > 1 - M$	$-\infty < \omega \leqslant \hat{\omega}$	λ_M
		$\hat{\omega} \leqslant \omega < -1/M$	λ_m

TABLE 3

Condition		ω-Domain	$\bar{\lambda}$
$M > 1$	$m < 1 - M$	$-1/m < \omega \leqslant \hat{\omega}$	λ_M
		$\hat{\omega} \leqslant \omega < +\infty$	λ_m
	$m \geqslant 1 - M$	$-1/m < \omega < +\infty$	λ_M

Subcase (i): $m < 1 - M$. Evidently, here (3.12) holds with $\hat{\omega} > -1/m$; hence

$$\bar{\lambda} = \begin{cases} \lambda_M & \text{if} \quad -1/m < \omega \leqslant \hat{\omega}, \\ \lambda_m & \text{if} \quad \hat{\omega} \leqslant \omega < +\infty. \end{cases} \tag{3.20}$$

Subcase (ii): $m \geqslant 1 - M$. For this case we have $\operatorname{sign}(\lambda_M - \lambda_m) = \operatorname{sign}(\hat{\omega} - \omega)$, where $\hat{\omega} < 0$; hence [see (3.17)]

$$\bar{\lambda} = \lambda_M.$$

The results of the two subcases are summarized in Table 3.

Finally, by combining (3.14), (3.19), and Tables 2, 3, we readily obtain the convergence ranges for the parameters τ, ω which are summarized in Table 1. ■

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1, *if* A *is a consistently ordered matrix, then the* Eω-DOJ *method converges iff* ω *and* τ *lie in the corresponding domains of Table* 4 *for* $M > 1$ *or* $M < 1$.

Proof. If A is a consistently ordered matrix, then [15]

$$M = -m = S(B). \tag{3.21}$$

TABLE 4

Conditions	Case	ω-Domain	τ-Domain[a]
$M < 1$	I	$-1/M < \omega \leqslant 1$	$0 < \tau < 2/\lambda_{-M}$
	II	$1 \leqslant \omega < 1/M$	$0 < \tau < 2/\lambda_M$
$M > 1$	III	$-\infty < \omega < -1/M$	$0 < \tau < 2/\lambda_{-M}$
	IV	$1/M < \omega \leqslant 1$	$2/\lambda_M < \tau < 0$
	V	$1 \leqslant \omega < +\infty$	$2/\lambda_{-M} < \tau < 0$

[a] $\lambda_{-M} \equiv \lambda(-M) = (1 - \omega M)(1 - M)$.

Next, we consider the two cases I and II as they were distinguished in the proof of Theorem 3.1.

Case I: Evidently, in view of (3.21), (3.7) is equivalent to equivalent to either of the following set of inequalities:

$$-\frac{1}{M} < \omega < \frac{1}{M} \quad \text{and} \quad M < 1, \tag{3.22}$$

or

$$-\infty < \omega < -\frac{1}{M} \quad \text{and} \quad M > 1. \tag{3.23}$$

Moreover, (3.10) yields

$$\bar{\lambda} = \max\{\lambda_M, \lambda_{-M}\}, \tag{3.24}$$

where

$$\lambda_{-M} \equiv \lambda(-M), \tag{3.25}$$

which in turn becomes

$$\bar{\lambda} = \begin{cases} \lambda_M & \text{if} \quad \omega \geqslant 1, \\ \lambda_{-M} & \text{if} \quad \omega \leqslant 1. \end{cases} \tag{3.26}$$

By combining (3.26) with (3.22) and (3.23), respectively, we see the validity of cases I, II, and III of Table 4.

Case II: Similarly, in this case $\lambda(M) < 0$ and $\lambda(-M) < 0$ are equivalent to

$$\frac{1}{M} < \omega < +\infty \quad \text{and} \quad M > 1, \tag{3.27}$$

whereas $\bar{\lambda}$ is defined by (3.18) and is given by

$$\bar{\lambda} = \begin{cases} \lambda_{-M} & \text{if} \quad \omega \geqslant 1, \\ \lambda_M & \text{if} \quad \omega \leqslant 1. \end{cases} \tag{3.28}$$

Evidently, if (3.28) is combined with (3.27), we verify cases IV and V of Table 4. ■

In order to increase the rate of convergence of the Eω-DOJ method we will determine the optimum values τ_0, ω_0 of τ, ω, respectively, in the sense of minimizing $S(\Gamma_{\tau,\omega})$. As is known [7, 10], $S(\Gamma_{\tau,\omega})$ is minimized when τ attains, the value

$$\tau_0 = \frac{2}{\bar{\lambda} + \underline{\lambda}}, \tag{3.29}$$

where

$$\bar{\lambda} = \begin{cases} \max\limits_{m \leqslant \mu \leqslant M} \lambda & \text{if} \quad \lambda > 0, \\ \min\limits_{m \leqslant \mu \leqslant M} \lambda & \text{if} \quad \lambda < 0, \end{cases} \qquad \underline{\lambda} = \begin{cases} \min\limits_{m \leqslant \mu \leqslant M} \lambda & \text{if} \quad \lambda > 0, \\ \max\limits_{m \leqslant \mu \leqslant M} \lambda & \text{if} \quad \lambda < 0, \end{cases} \tag{3.30}$$

and its corresponding value is given by the expression

$$S(\Gamma_{\tau_0,\omega}) = \frac{|P(\mathscr{B}_\omega) - 1|}{P(\mathscr{B}_\omega) + 1} \qquad \text{with} \quad P(\mathscr{B}_\omega) = \frac{\bar{\lambda}}{\underline{\lambda}}. \tag{3.31}$$

In addition, we note that if $\lambda > 0$ ($\lambda < 0$), then $P(\mathscr{B}_\omega) > 1$ (< 1) and $S(\Gamma_{\tau,\omega})$ is an increasing (decreasing) function of $P(\mathscr{B}_\omega)$. Therefore, for a further reduction of $S(\Gamma_{\tau_0,\omega})$ we have to determine the optimum value of ω such that $P(\mathscr{B}_\omega)$ is minimized (maximized).

THEOREM 3.3. *Let A be a matrix with nonvanishing diagonal elements such that B has two real multiple eigenvalues m, M with $m < M$, where either $M < 1$ or $M > 1$ and $m \neq 1 - M$. Then, for*

$$\omega_0 = \frac{1}{1 - M - m} \quad \text{and} \quad \tau_0 = \frac{1}{1 + \omega_0 Mm}, \tag{3.32}$$

we have

$$S(\Gamma_{\tau_0,\omega_0}) = 0. \tag{3.33}$$

TABLE 5

Conditions		Case	ω-domain	$\bar{\lambda}$	$\underline{\lambda}$	$P(\mathscr{B}_\omega)$
$M<1$		I	$-1/M<\omega\leqslant\hat{\omega}$	λ_m	λ_M	λ_m/λ_M
		II	$\hat{\omega}\leqslant\omega<-1/m$	λ_M	λ_m	λ_M/λ_m
		III	$-\infty<\omega<-1/M$	λ_m	λ_M	λ_m/λ_M
$M>1$	$m<1-M$	IV	$-\infty<\omega<-1/M$	λ_m	λ_M	λ_m/λ_M
		V	$-1/m<\omega\leqslant\hat{\omega}$	λ_M	λ_m	λ_M/λ_m
		VI	$\hat{\omega}\leqslant\omega<+\infty$	λ_m	λ_M	λ_m/λ_M
	$m>1-M$	VII	$-\infty<\omega\leqslant\hat{\omega}$	λ_M	λ_m	λ_M/λ_m
		VIII	$\hat{\omega}\leqslant\omega<-1/M$	λ_m	λ_M	λ_m/λ_M
		IX	$-1/m<\omega<+\infty$	λ_M	λ_m	λ_M/λ_m

Proof. For the determination of $P(\mathscr{B}_\omega)$, for the different ranges of ω (see Table 1) we need from (3.31) the expression of $\underline{\lambda}$. By following the analysis of the proof of Theorem 3.1 we can easily find the expressions of $P(\mathscr{B}_\omega)$ given in Table 5.
But

$$\operatorname{sign}\left[\frac{\partial}{\partial\omega}\left(\frac{\lambda_M}{\lambda_m}\right)\right]=\operatorname{sign}(1-M), \tag{3.34}$$

which implies that independent of the conditions on M and m, $P(\mathscr{B}_\omega)$ is minimized for

$$\omega_0=\hat{\omega}. \tag{3.35}$$

For this value of ω, (3.30) (see also Table 5) yields

$$\bar{\lambda}=\underline{\lambda}=1+\omega_0 Mm. \tag{3.36}$$

Hence (3.29) yields the value of τ_0 given by (3.32), whereas because of (3.31), (3.33) follows, since $P(\mathscr{B}_{\omega_0})=1$. ∎

THEOREM 3.4. *Under the hypothesis of Theorem* 3.3, *if A is a consistently ordered matrix, then the Eω-DOJ method degenerates into the extrapolated double Jacobi (EDJ) method when ω attains its optimum value.*

Proof. Because of the validity of (3.21) and following the proof of Theorem 3.2, we find the expressions for $P(\mathscr{B}_\omega)$ presented in Table 6 for the

TABLE 6

Condition	ω-domain	$\bar{\lambda}$	$\underline{\lambda}$	$P(\mathscr{B}_\omega)$
$M<1$	$-1/M<\omega\leqslant 1$	λ_{-M}	λ_M	λ_{-M}/λ_M
	$1\leqslant\omega<1/M$	λ_M	λ_{-M}	λ_M/λ_{-M}
$M>1$	$-\infty<\omega-1/M$	λ_{-M}	λ_M	λ_{-M}/λ_M
	$1/M<\omega\leqslant 1$	λ_M	λ_{-M}	λ_M/λ_{-M}
	$1\leqslant\omega<+\infty$	λ_{-M}	λ_M	λ_{-M}/λ_M

different ranges of the parameter ω. Since A is a consistently ordered matrix, it follows that $m=-M$, in which case (3.34) yields

$$\operatorname{sign}\left[\frac{\partial}{\partial\omega}\left(\frac{\lambda_M}{\lambda_{-M}}\right)\right]=\operatorname{sign}(1-M). \tag{3.37}$$

Therefore, when $M<1$ it follows that λ_M/λ_{-M} is a decreasing function of ω for $-1/M<\omega\leqslant 1$ (see Table 6), whereas for $1\leqslant\omega<1/M$ it decreases with ω. This shows that the minimum value of $P(\mathscr{B}_\omega)$ will be achieved at

$$\omega_0=1. \tag{3.38}$$

A similar argument holds when $M>1$. ■

4. FINAL REMARKS AND CONCLUSIONS

The fact that one J-iteration can be carried out in parallel was the primary motive of expressing the inverse of A in terms of B, thus letting the conditioning matrix of (1.6) be a "good" approximation to A^{-1}. Since the updated iterates in (1.7) are obtained, in essence, from the repeated use of the J-method, a parallel implementation is feasible. In an attempt to tackle the problem in its general form, we analysed its convergence and concluded (see Section 2) that for practical purposes the best approximation for A^{-1} is to keep only the first two terms in its Neumann series expansion. Since we have kept the computational work in our iterative scheme reasonably low, we introduced an additional real parameter ω to increase its convergence rate further, thus deriving the Eω-DOJ method. Under our basic assumption that the matrix B has 2 real eigenvalues, we presented the convergence ranges of the parameters involved in Table 1, which evidently depend upon the

position of m and M. Finally, the optimum values of these parameters are determined, and proved that the spectral radius of the Eω-DOJ's iteration matrix is reduced to zero when $M < 1$ or $M > 1$ and $m \neq 1 - M$. From Theorem 3.4 we observe that when A is a consistently ordered matrix, the Eω-DOJ degenerates into the EDOJ, since $\omega_0 = 1$. Therefore, for the aforementioned class of matrices the EDOJ must be combined with the conjugate gradient [3, 8] or semi-iterative methods [14] to achieve fast convergence.

In order to verify our theory, let us consider the system $Au = b$, where $a_{ii} = 1$; $a_{ij} = s$, $i \neq j$; and b is chosen such that the solution is the vector $u = (1, 1, \ldots, 1)^T$. Since A is symmetric and possesses the eigenvalues $1 - s$ (of multiplicity $N - 1$) and $1 + (N - 1)s$ [12], it follows that if $0 < s < 1$, then A is a positive definite matrix, $M = s < 1$, and $m = (1 - N)s$. According to Theorem 3.3 the optimum value of ω is given by $\omega_0 = 1/(1 - 2s + Ns)$, and letting $s = 0.25$, we compute λ_M and λ_m from (3.5), thus finding

$$\lambda_M = \lambda_m = 0.81245,$$

verifying (3.33). On the other hand, we carried out a number of numerical calculations using the Eω-DOJ method for $s = 0.1(0.1)0.9$ and for the successive values $N = 20$, 40, 60, and 80. As a starting vector, $u^{(0)} = (0, 0, \ldots, 0)^T$ was used, and the iterations were terminated when the inequality $\|u^{(n+1)} - u^{(n)}\|_\infty \leqslant 10^{-6}$ was satisfied. In all our experiments the Eω-DOJ method converged in two iterations only, thus verifying our theoretical expectations.

For the implementation of the algorithm on existing parallel architectures of the type SIMD and MIMD (e.g. ICL-DAP, CRAY-1, NEPTUNE system, Loughborough) the following observations can be readily made.

Since the method is a Jacobi-type algorithm, the solution of the ith iteration depends totally on that of the $(i - 1)^{\text{th}}$ iteration. Thus, a synchronous version of this algorithm can be formulated for implementation on parallel systems of the MIMD, type where the solution of each iteration is stored in the shared memory and each processor evaluates $\lceil N/P \rceil$ components of the solution vector, where P is the number of identical processors at each iteration. The algorithm is synchronous: care should be taken to modify the solution in the shared memory after all the processors have completed their evaluation of the relevant points in a single iteration.

An asynchronous version of this algorithm can be envisaged whereby each processor on evaluating its assigned vector components uses the most recent values of the components evaluated by other processors and stores its results back into the shared memory as soon as it has evaluated a point (i.e. the chaotic Jacobi method; see [2], [1]). The components of the vector u in this

version can be considered as forming a queue, and each processor, on completing the evaluation of a point, takes the next available point to evaluate from this queue. A critical section must be used for the pointer pointing to the first available component of the vector u when it is being incremented by a processor, and when this pointer reaches the value of N it has to be reset to 1.

On the synchronous systems like the ICL-DAP or the CRAY-1, the implementation of the algorithm is fairly straightforward. The matrix-vector multiplications involved can be performed very efficiently on these systems. For problems arising from partial differential equations, where the resulting matrices are usually banded, certain advantages can be gained by using the special routines written for sparse matrix-vector multiplication on the above systems.

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